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Tensor operators for $\mathcal{U}_h(sl(2))$

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Abstract. Tensor operators for the Jordanian quantum algebra $\mathcal{U}_h(sl(2))$ are considered. Some explicit examples of them, which are obtained in the boson or fermion realization, are given and their properties are studied. It is also shown that the Wigner–Eckart theorem can be extended to $\mathcal{U}_h(sl(2))$.

1. Introduction

Recent studies on quantum matrices in two dimensions show that the Lie group $GL(2)$ admits two kinds of quantum deformation [1–3]. One of them is denoted by $GL_{p,q}(2)$ and has been studied extensively since the beginning of quantum group theory. The other is denoted by $GL_{g,h}(2)$ and is sometimes called the Jordanian quantum group. $SL_h(2)$ is the special case of $GL_{g,h}(2)$ obtained by setting $g = h$ and the quantum determinant to unity. The dual of $SL_h(2)$ is a deformation of the universal enveloping algebra of $sl(2)$ [4] and is called the Jordanian quantum algebra $\mathcal{U}_h(sl(2))$. The explicit form of the universal R-matrix for $\mathcal{U}_h(sl(2))$ is known [5, 6]. It is also known that $\mathcal{U}_h(sl(2))$ can be obtained from the Drinfeld–Jimbo $\mathcal{U}_q(sl(2))$ by a contraction [7]. The Hopf algebra dual to $GL_{g,h}(2)$ was found very recently [8].

The representation theory of $\mathcal{U}_h(sl(2))$ seems to have attracted some interest, since it has been revealed that the representation theories of $\mathcal{U}_h(sl(2))$ and $sl(2)$ have some similarities. Finite-dimensional irreducible representations (irreps) were first considered in [9], then a simple way to construct irreps with a nonlinear relation between the generators of $\mathcal{U}_h(sl(2))$ and $sl(2)$ was proposed [10]. They show that the finite-dimensional irreps of $\mathcal{U}_h(sl(2))$ can be classified in the same way as those of $sl(2)$ (see also [11]). The infinite-dimensional representations are considered in [12] with boson realizations. The first attempt to decompose a tensor product of two finite-dimensional irreps was made in [13], then the problem was completely solved in [14, 15]. This gives another similarity between the representation theories of $\mathcal{U}_h(sl(2))$ and $sl(2)$, that is, the decomposition rule is exactly the same as for $sl(2)$. Furthermore, an explicit formula for $\mathcal{U}_h(sl(2))$ Clebsch–Gordan coefficients (CGCs) is given in [14].

The nonlinear relation introduced in [10] gives an interesting observation for the coproduct of $\mathcal{U}_h(sl(2))$. We can regard $\mathcal{U}_h(sl(2))$ as the angular momentum algebra with a non-standard coupling rule. This might suggest that $\mathcal{U}_h(sl(2))$ has lots of applications to various fields in physics.

In this paper, we further develop the representation theory of $\mathcal{U}_h(sl(2))$, in particular tensor operators will be studied. We review the known results on the representation of $\mathcal{U}_h(sl(2))$ in the next two sections, in order to fix our notation and to list formulae used in the subsequent sections. Tensor operators for $\mathcal{U}_h(sl(2))$ are introduced in section 4 according to [16]. Some explicit examples of $\mathcal{U}_h(sl(2))$ tensor operators are given and their properties are considered. In section 5, we consider an extension of the Wigner–Eckart theorem to $\mathcal{U}_h(sl(2))$.

2. $\mathcal{U}_h(sl(2))$ and its representations

The Jordanian quantum algebra $\mathcal{U}_h(sl(2))$ is an associative algebra with unity, and is generated by X, Y and H subjected to the relations [4]

$$[X, Y] = H \qquad [H, X] = 2\frac{\sinh hX}{h}$$
$$[H, Y] = -Y(\cosh hX) - (\cosh hX)Y \tag{2.1}$$

where h is the deformation parameter. The coproduct Δ, the counit ϵ and the antipode S are given by

$$\Delta(X) = X \otimes 1 + 1 \otimes X$$
$$\Delta(Y) = Y \otimes \mathrm{e}^{hX} + \mathrm{e}^{-hX} \otimes Y \tag{2.2}$$
$$\Delta(H) = H \otimes \mathrm{e}^{hX} + \mathrm{e}^{-hX} \otimes H$$
$$\epsilon(X) = \epsilon(Y) = \epsilon(H) = 0 \tag{2.3}$$
$$S(X) = -X \qquad S(Y) = -\,\mathrm{e}^{hX} Y\, \mathrm{e}^{-hX} \qquad S(H) = -\,\mathrm{e}^{hX} H\, \mathrm{e}^{-hX} \tag{2.4}$$

so that $\mathcal{U}_h(sl(2))$ is a Hopf algebra. The Casimir element belonging to the centre of $\mathcal{U}_h(sl(2))$ is [5]

$$C = \frac{1}{2h}\{Y(\sinh hX) + (\sinh hX)Y\} + \frac{1}{4}H^2 + \frac{1}{4}(\sinh hX)^2. \tag{2.5}$$

The Jordanian quantum algebra $\mathcal{U}_h(sl(2))$ is a non-standard deformation of the universal enveloping algebra of $sl(2)$, since all expressions given in (2.1)–(2.5) reduce to the corresponding ones for $sl(2)$ in the limit of $h \to 0$.

Note that we can eliminate the deformation parameter h from all expressions by making the replacement $hX \to X$ and $h^{-1}Y \to Y$. Thus, $\mathcal{U}_h(sl(2))$ is isomorphic to $\mathcal{U}_{h=1}(sl(2))$. We, however, keep the parameter h throughout this paper in order to consider the limit of $h \to 0$.

The finite-dimensional irreps of $\mathcal{U}_h(sl(2))$ can be obtained by using the nonlinear relation between generators of $\mathcal{U}_h(sl(2))$ and $sl(2)$ [10]. With the definition

$$Z_+ = \frac{2}{h}\tanh\frac{hX}{2} \qquad Z_- = \left(\cosh\frac{hX}{2}\right) Y \left(\cosh\frac{hX}{2}\right) \tag{2.6}$$

it follows that $Z_\pm$ and H satisfy the $sl(2)$ commutation relations

$$[H, Z_\pm] = \pm 2Z_\pm \qquad [Z_+, Z_-] = H \tag{2.7}$$

and the Casimir element (2.5) is rewritten as

$$C = Z_+Z_- + \frac{H}{2}\left(\frac{H}{2} - 1\right). \tag{2.8}$$

We can take undeformed representation bases for $Z_\pm$ and H

$$Z_\pm|jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\,|jm \pm 1\rangle$$
$$H|jm\rangle = 2m|jm\rangle \tag{2.9}$$
$$C|jm\rangle = j(j+1)|jm\rangle$$

where $j = 0, 1/2, 1, 3/2, \ldots$ and $m = -j, -j+1, \ldots, j$. The vectors $\{|jm\rangle\}$ are nothing but the representation bases for $sl(2)$; their complete orthonormality and the representation matrices for bra vectors follow immediately. The representation matrices for X and Y can be obtained by solving (2.9) with respect to $Z_\pm$. The closed form of their expressions is given in [14] and this shows that finite-dimensional, highest-weight irreps for $\mathcal{U}_h(sl(2))$ are classified in the same way as for $sl(2)$.

3. Clebsch–Gordan coefficients for $\mathcal{U}_h(sl(2))$

In this section, we review some known results on the tensor products of two irreps given in the previous section. Although $Z_\pm$ and H are the elements of $sl(2)$, their coproducts are given in terms of $\Delta(X)$, $\Delta(Y)$ and $\Delta(H)$ (see [13, 14] for explicit formulae of $\Delta(Z_\pm)$) so that the irreducible decomposition of tensor product representations is a non-trivial problem. This problem is solved in [13–15].

Theorem 1. The tensor product of two irreps of $\mathcal{U}_h(sl(2))$ with highest weight j_1 and j_2 is completely reducible and the decomposition into irreps is given by

$$j_1 \otimes j_2 = j_1 + j_2 \oplus j_1 + j_2 - 1 \oplus \cdots \oplus |j_1 - j_2| \tag{3.1}$$

where each irrep is multiplicity free. Namely, the decomposition rules for $\mathcal{U}_h(sl(2))$ and $sl(2)$ are the same.

The CGCs for $\mathcal{U}_h(sl(2))$ can be obtained by introducing new vectors defined by

$$|(j_1m_1)(j_2m_2)\rangle = \sum_{k_i=m_i}^{j_i} \alpha_{k_1,k_2}^{m_1,m_2} |j_1k_1\rangle \otimes |j_2k_2\rangle \tag{3.2}$$

where the coefficients $\alpha_{k_1,k_2}^{m_1,m_2}$ are given by

$$\alpha_{k_1,k_2}^{m_1,m_2} = (-1)^{k_2-m_2} \frac{h}{2}^{k_1+k_2-m_1-m_2} D_{k_1,k_2}^{m_1,m_2} (b_{k_1,k_2}^{m_1,m_2} - b_{k_1-1,k_2-1}^{m_1,m_2}) \tag{3.3}$$

with

$$D_{k_1,k_2}^{m_1,m_2} = \left\{ \frac{(j_1-m_1)!(j_1+k_1)!(j_2-m_2)!(j_2+k_2)!}{(j_1+m_1)!(j_1-k_1)!(j_2+m_2)!(j_2-k_2)!} \right\}^{1/2}$$

$$b_{k_1,k_2}^{m_1,m_2} = \binom{m_1+k_1}{k_2-m_2} \binom{m_2+k_2}{k_1-m_1}.$$

We use the following definition of the binomial coefficients, since the superscripts m_i in $b_{k_1,k_2}^{m_1,m_2}$ take negative values

$$\binom{n}{m} = \begin{cases} \dfrac{n(n-1)(n-2)\cdots(n-m+1)}{m!} & \text{for } m \geqslant 0 \\ 0 & \text{for } m < 0. \end{cases}$$

Note that the coefficients $\alpha_{k_1,k_2}^{m_1,m_2}$ depend on j_1 and j_2, although the dependence is not shown explicitly. Note also that, in the limit of $h \to 0$, all coefficients $\alpha_{k_1,k_2}^{m_1,m_2}$ vanish except for $\alpha_{m_1,m_2}^{m_1,m_2} = 1$ so that $|(j_1m_1)(j_2m_2)\rangle \to |j_1m_1\rangle \otimes |j_2m_2\rangle$. We refer to the vectors (3.2) as 'intermediate vectors' in this paper, since they are an intermediate step to the CGCs.

The important property of the intermediate vectors, which plays a crucial role in the following discussion, is the action of $\Delta(Z_\pm)$ and $\Delta(H)$ on the intermediate vectors given by

$$\begin{aligned}
\Delta(H)|(j_1m_1)(j_2m_2)\rangle &= 2(m_1+m_2)|(j_1m_1)(j_2m_2)\rangle \\
\Delta(Z_\pm)|(j_1m_1)(j_2m_2)\rangle &= \sqrt{(j_1\mp m_1)(j_1\pm m_1+1)}|(j_1m_1\pm 1)(j_2m_2)\rangle \\
&\quad +\sqrt{(j_1\mp m_2)(j_2\pm m_2+1)}|(j_1m_1)(j_2m_2\pm 1)\rangle. \qquad (3.4)
\end{aligned}$$

This tells us that the action of $\Delta(Z_\pm)$ and $\Delta(H)$ on an intermediate vector is the same as the action of the undeformed coproducts of $sl(2)$ elements on a vector $|j_1m_1\rangle\otimes|j_2m_2\rangle$. Therefore, the bases of irreps for $\Delta(Z_\pm)$ and $\Delta(H)$ are obtained by linear combinations of the intermediate vectors with the CGCs for $sl(2)$:

$$\begin{aligned}
|jm\rangle &= \sum_{m_1+m_2=m} C^{j_1,j_2,j}_{m_1,m_2,m}|(j_1m_1)(j_2m_2)\rangle \\
&= \sum_{k_i=m_i}^{j_i}\sum_{m_1+m_2=m} C^{j_1,j_2,j}_{m_1,m_2,m}\alpha^{m_1,m_2}_{k_1,k_2}|j_1k_1\rangle\otimes|j_2k_2\rangle \qquad (3.5)
\end{aligned}$$

where $C^{j_1,j_2,j}_{m_1,m_2,m}$ is an $sl(2)$ CGC.

The orthogonality of the coefficients $\alpha^{m_1,m_2}_{k_1,k_2}$ is obtained in [15]

$$\sum_{k_1,k_2}\alpha^{m_1,m_2}_{k_1,k_2}\alpha^{-n_1,-n_2}_{-k_1,-k_2} = \delta_{m_1,n_1}\delta_{m_2,n_2}. \qquad (3.6)$$

Before closing this section, we add a new result. The intermediate vectors for the dual space (space spanned by bra vectors) are given by

$$\langle(j_1m_1)(j_2m_2)| = \sum_{k_i=-j_i}^{m_i}\alpha^{-m_1,-m_2}_{-k_1,-k_2}\langle j_1k_1|\otimes\langle j_2k_2|. \qquad (3.7)$$

The action of $\Delta(Z_\pm)$ and $\Delta(H)$ on (3.7) is the same as the action of the undeformed coproducts of $sl(2)$ elements on a vector $\langle j_1m_1|\otimes\langle j_2m_2|$. This can be proved by the same method as in [14]. The orthogonality of the coefficients $\alpha^{m_1,m_2}_{k_1,k_2}$ results in the orthonormality of the intermediate vectors

$$\langle(j_1n_1)(j_2n_2)|(j_1m_1)(j_2m_2)\rangle = \delta_{n_1,m_1}\delta_{n_2,m_2}. \qquad (3.8)$$

Note that the representations of $\Delta(H)$ and $\Delta(Z_\pm)$ on the intermediate vectors (for both bra and ket vectors) are unitary. Therefore, equation (3.8) is nothing but the well known fact that the eigenvectors of a hermitian operator with different eigenvalues are orthogonal to each other.

4. Tensor operators for $\mathcal{U}_h(sl(2))$

4.1. Definition of tensor operators

Rittenberg and Scheunert [16] gave a general definition of tensor operators for a Hopf algebra. To define tensor operators, we first define the adjoint action of a Hopf algebra.

Definition 1. Let $\mathcal{H}$ be a Hopf algebra, let W, W' be its representation space, and let t be an operator which carries W into W'. Then the adjoint action of $c\in\mathcal{H}$ on t is defined by

$$\mathrm{ad}\, c(t) = \sum_i c_i t S(c_i') \qquad (4.1)$$

where the coproduct for c is written as $\Delta(c)=\sum_i c_i\otimes c_i'$.

The adjoint action has two important properties

$$\mathrm{ad}\, cc'(t) = \mathrm{ad}\, c \circ \mathrm{ad}\, c'(t)$$
$$\mathrm{ad}\, c(t \otimes s) = \sum_i (\mathrm{ad}\, c_i(t)) \otimes (\mathrm{ad}\, c_i'(s)).$$

From these properties, we can show that the adjoint action gives a representation of $\mathcal{H}$

$$\mathrm{ad}[c, c'](t) = [\mathrm{ad}\, c, \mathrm{ad}\, c'](t). \tag{4.2}$$

Tensor operators are defined as operators which form representation bases of a Hopf algebra under the adjoint action.

Definition 2. Let T be a set of operators, and $D(c)^{(j)}$ be a representation matrix of $c \in \mathcal{H}$ with the highest weight j. The operators $t_{jm} \in T$ are called rank j tensor operators, if they satisfy the relations

$$\mathrm{ad}\, c(t_{jm}) = \sum_k D(c)^{(j)}_{km} t_{jk}. \tag{4.3}$$

The adjoint action of X, Y and H is given by

$$\begin{aligned} \mathrm{ad}\, X(t_{jm}) &= [X, t_{jm}] \\ \mathrm{ad}\, Y(t_{jm}) &= \mathrm{e}^{-hX}[\mathrm{e}^{hX} Y, t_{jm}]\, \mathrm{e}^{-hX} \\ \mathrm{ad}\, H(t_{jm}) &= \mathrm{e}^{-hX}[\mathrm{e}^{hX} H, t_{jm}]\, \mathrm{e}^{-hX}. \end{aligned} \tag{4.4}$$

4.2. Some examples of $\mathcal{U}_h(sl(2))$ tensor operators

In this section, we shall give explicit expressions of three kinds of $\mathcal{U}_h(sl(2))$ tensor operators. To show the existence of $\mathcal{U}_h(sl(2))$ tensor operators, it is enough to construct rank-$\frac{1}{2}$ tensor operators, since higher-rank tensor operators can be obtained by decomposing a tensor product of some rank-$\frac{1}{2}$ tensor operators. This is due to the fact that tensor operators are representation bases of $\mathcal{U}_h(sl(2))$ and we have an explicit formula for the $\mathcal{U}_h(sl(2))$ CGCs.

The tensor operators given here are: (i) rank-$\frac{1}{2}$ tensor operators in the fermion realization of $\mathcal{U}_h(sl(2))$; (ii) rank-$\frac{1}{2}$ tensor operators in the boson realization of $\mathcal{U}_h(sl(2))$; (iii) rank-1 tensor operators constructed by the generators of $\mathcal{U}_h(sl(2))$ themselves. The basic idea for (i) and (ii) is quite simple. We realize $\mathcal{U}_h(sl(2))$ with the generators of $sl(2)$

$$H = J_0 \qquad X = \frac{2}{h}\mathrm{arctanh}\left(\frac{h}{2}J_+\right) \qquad Y = \sqrt{1-\left(\frac{h}{2}J_+\right)^2} J_- \sqrt{1-\left(\frac{h}{2}J_+\right)^2} \tag{4.5}$$

where $J_\pm$ and J_0 are generators of $sl(2)$. This is obtained by solving (2.6) with respect to X and Y and regarding $\{Z_\pm, H\}$ as the generators of $sl(2)$. Then we realize $sl(2)$ in terms of fermions or bosons. We need representation matrices of X, Y and H for $j = 1/2$ and 1 to find the rank-$\frac{1}{2}$ or rank-1 tensor operators. The representation matrices for $j = 1/2$ read

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and for $j = 1$

$$X = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -h^2/2\sqrt{2} & 0 \\ \sqrt{2} & 0 & -h^2/2\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$
$$H = \mathrm{diag}(2, 0, -2).$$

Note that the representation matrices for $j = 1/2$ are the same as those for $h \to 1$; however, rank-$\frac{1}{2}$ tensor operators are non-trivial since the adjoint action has a different form (see (4.4)).

Let us first consider the fermion realization. We introduce two kinds of mutually anticommuting fermions

$$\{a_i, a_j^\dagger\} = \delta_{ij} \qquad \{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0 \qquad i, j = 1, 2.$$

These fermions realize $sl(2)$ (the so-called fermion quasi-spin formalism),

$$J_+ = a_1^\dagger a_2^\dagger \qquad J_- = a_2 a_1 \qquad J_0 = N_1 + N_2 - 1 \tag{4.6}$$

where $N_i \equiv a_i^\dagger a_i$ is the number operator for the ith fermion. This realization gives two representations of $sl(2)$ and $\mathcal{U}_h(sl(2))$. One of them is the two-dimensional irrep whose representation space $W^{(1/2)}$ has bases $|\frac{1}{2}\ \frac{1}{2}\rangle = a_1^\dagger a_2^\dagger|0\rangle$ and $|\frac{1}{2}\ -\frac{1}{2}\rangle = |0\rangle$, where $|0\rangle$ denotes the fermion vacuum. The other is the trivial representation whose representation space $W^{(0)}$ is spanned by $a_1^\dagger|0\rangle$ or $a_2^\dagger|0\rangle$. The advantage of the fermions is that the adjoint action has a simpler form, since the nilpotency of fermions results in $X^2 = 0$. We find two kinds of rank-$\frac{1}{2}$ tensor operators in this realization

$$t_{1/2\,1/2} = -a_1^\dagger \qquad t_{1/2\,-1/2} = -a_2 + h(N_2 - 1)a_1^\dagger \tag{4.7}$$

and

$$t_{1/2\,1/2} = a_2^\dagger, t_{1/2\,-1/2} = -a_1 - h(N_1 - 1)a_2^\dagger. \tag{4.8}$$

Straightforward computation shows that these satisfy the definition of rank-$\frac{1}{2}$ tensor operators. It is also easy to see that the action of both (4.7) and (4.8) on $W^{(1/2)}$ results in $W^{(0)}$ and *vice versa*.

Next we consider the boson realization. With two kinds of mutually commuting bosons

$$[b_i, b_j^\dagger] = \delta_{ij} \qquad [b_i, b_j] = [b_i^\dagger, b_j^\dagger] = 0 \qquad i, j = 1, 2$$

the Lie algebra $sl(2)$ is realized as (the Jordan–Schwinger realization)

$$J_+ = b_1^\dagger b_2 \qquad J_- = b_2^\dagger b_1 \qquad J_0 = N_1 - N_2 \tag{4.9}$$

where $N_i = b_i^\dagger b_i$ is the number operator for the ith boson. We obtain any irrep of $sl(2)$ and $\mathcal{U}_h(sl(2))$ in this realization. Let us denote the representation space for highest weight j by $W^{(j)}$, then the bases of $W^{(j)}$ are given by

$$|jm\rangle = \frac{(b_1^\dagger)^{j+m}(b_2^\dagger)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|0\rangle \qquad m = -j, -j+1, \ldots, j \tag{4.10}$$

where $|0\rangle$ denotes the boson vacuum. It is shown, by straightforward computation, that there exist two kinds of rank-$\frac{1}{2}$ tensor operators in this realization

$$t_{1/2\,1/2} = \left(1 - \frac{h}{2}J_+\right)^{-1} b_1^\dagger \qquad t_{1/2\,-1/2} = \left(1 - \frac{h}{2}J_+\right) b_2^\dagger + \frac{h}{2}(t_{1/2\,1/2} - b_1^\dagger J_0) \tag{4.11}$$

and

$$t_{1/2\,1/2} = -\left(1 - \frac{h}{2}J_+\right)^{-1} b_2 \qquad t_{1/2\,-1/2} = \left(1 - \frac{h}{2}J_+\right) b_1 + \frac{h}{2}(t_{1/2\,1/2} + b_2 J_0). \tag{4.12}$$

The action of (4.11) on $W^{(j)}$ reads

$$t_{1/2\,1/2}|jm\rangle = \sum_{n=0}^{j-m}\left(\frac{h}{2}\right)^n \Gamma_n^{jm}\left|j+\frac{1}{2}\ m+\frac{1}{2}+n\right\rangle \tag{4.13}$$

$$\begin{aligned} t_{1/2\,-1/2}|jm\rangle &= \sqrt{j-m+1}\left|j+\frac{1}{2}\ m-\frac{1}{2}\right\rangle - \frac{h}{2}(j+m)\sqrt{j+m+1}\left|j+\frac{1}{2}\ m+\frac{1}{2}\right\rangle \\ &\quad +\frac{h}{2}\sum_{n=1}^{j-m}\left(\frac{h}{2}\right)^n \Gamma_n^{jm}\left|j+\frac{1}{2}\ m+\frac{1}{2}+n\right\rangle \end{aligned} \tag{4.14}$$

where

$$\Gamma_n^{jm} = \left\{\frac{(j-m)!(j+m+n+1)!}{(j+m)!(j-m-n)!}\right\}^{1/2}.$$

On the other hand, the action of (4.12) on $W^{(j)}$ reads

$$t_{1/2\,1/2}|jm\rangle = -\sum_{n=0}^{j-m-1}\left(\frac{h}{2}\right)^n \Lambda_n^{jm}\left|j-\frac{1}{2}\ m+\frac{1}{2}+n\right\rangle \tag{4.15}$$

$$\begin{aligned} t_{1/2\,-1/2}|jm\rangle &= \left|j-\frac{1}{2}\ m-\frac{1}{2}\right\rangle - \frac{h}{2}\sqrt{j-m}(j-m-1)\left|j-\frac{1}{2}\ m+\frac{1}{2}+n\right\rangle \\ &\quad -\frac{h}{2}\sum_{n=1}^{j-m-1}\left(\frac{h}{2}\right)^n \Lambda_n^{jm}\left|j-\frac{1}{2}\ m+\frac{1}{2}+n\right\rangle \end{aligned} \tag{4.16}$$

where

$$\Lambda_n^{jm} = \left\{\frac{(j-m)!(j+m+n)!}{(j+m)!(j-m-n-1)!}\right\}^{1/2}.$$

Therefore, we see that the action of tensor operators (4.11) gives rise to a mapping $W^{(j)} \to W^{(j+1/2)}$, while the tensor operators (4.12) give rise to $W^{(j)} \to W^{(j-1/2)}$.

The third example of tensor operators is constructed with the generators of $\mathcal{U}_h(sl(2))$ themselves. It is also straightforward to verify that the rank-1 tensor operators are given by

$$\begin{aligned} t_{1\,1} &= -\mathrm{e}^{hX}\frac{\sinh hX}{h} \\ t_{1\,0} &= \frac{\mathrm{e}^{hX}H}{\sqrt{2}} \\ t_{1\,-1} &= \mathrm{e}^{-hX/2}Y\,\mathrm{e}^{-hX/2} + \frac{h}{2}\,\mathrm{e}^{hX/2}H\,\mathrm{e}^{hX/2} - \frac{h}{2}H^2. \end{aligned} \tag{4.17}$$

These are a combination of the $\mathcal{U}_h(sl(2))$ generators so that they can act on any irrep space and do not change the value of highest weight: $t_{1\,m} : W^{(j)} \to W^{(j)}$.

All the results given here are a natural analogue of $sl(2)$, since they have counterparts, which are well known properties of the $sl(2)$ tensor operators, in the limit of $h \to 0$. Therefore, we have seen new similarities between the representation theories of $\mathcal{U}_h(sl(2))$ and $sl(2)$.

5. The Wigner–Eckart theorem

The results in the previous section enable us to consider an extension of the Wigner–Eckart theorem to the Jordanian quantum algebra $\mathcal{U}_h(sl(2))$. The purpose of this section is to show that the Wigner–Eckart theorem can be extended to $\mathcal{U}_h(sl(2))$.

Theorem 2. Let $T^{(j_1)}$ be a set of rank j_1 tensor operators, let $W^{(j)}$ be an irrep space of $\mathcal{U}_h(sl(2))$ with highest weight j and suppose that $t_{j_1 m_1} \in T^{(j_1)} : W^{(j_2)} \to W^{(j)}$. Then

$$\langle jm|t_{j_1m_1}|j_2m_2\rangle = I(j_1j_2j)\sum_{n_i=-j_i}^{j_i} \alpha^{-n_1,-n_2}_{-m_1,-m_2} C^{j_1,j_2,j}_{n_1,n_2,m} \tag{5.1}$$

where $I(j_1j_2j)$ is a constant independent of m_1, m_2 and m.

Proof. According to [17], we consider an element $t_{j_1m_1} \otimes |j_2m_2\rangle$ of $T^{(j_1)} \otimes W^{(j_2)}$. Both $T^{(j_1)}$ and $W^{(j_2)}$ are representation spaces of $\mathcal{U}_h(sl(2))$ so that $\Delta(c)$, $c \in \mathcal{U}_h(sl(2))$, acts on $T^{(j_1)} \otimes W^{(j_2)}$. For example,

$$\Delta(H)t_{j_1m_1} \otimes |j_2m_2\rangle = \mathrm{ad}\, H(t_{j_1m_1}) \otimes \mathrm{e}^{hX}|j_2m_2\rangle + \mathrm{ad}\,\mathrm{e}^{-hX}(t_{j_1m_1}) \otimes H|j_2m_2\rangle. \tag{5.2}$$

The left-hand side of $\otimes$ is a tensor operator, since the adjoint action is a linear transformation on $T^{(j_1)}$. Thus we can consider an action of the left-hand side of $\otimes$ on the right-hand side: $t \otimes |jm\rangle \to t|jm\rangle$. This operation is called a 'contraction' in [17]. Noting that $\mathrm{ad}\,\mathrm{e}^{-hX}(t_{j_1m_1}) = \mathrm{e}^{-hX}t_{j_1m_1}\mathrm{e}^{hX}$ and contracting (5.2), we obtain $Ht_{j_1m_1}|j_2m_2\rangle$. A similar calculation shows that

$$\Delta(c)t_{j_1m_1} \otimes |j_2m_2\rangle \to ct_{j_1m_1}|j_2m_2\rangle \tag{5.3}$$

where the arrow means that the right-hand side is a result of the contraction.

An intermediate vector on $T^{(j_1)} \otimes W^{(j_2)}$ is given by

$$|(j_1m_1)(j_2m_2)\rangle = \sum_{k_i=-j_i}^{j_i} \alpha^{m_1,m_2}_{k_1,k_2} t_{j_1k_1} \otimes |j_2k_2\rangle. \tag{5.4}$$

Because of (3.4), the action of $\Delta(Z_\pm)$ on the vector yields

$$\begin{aligned}\Delta(Z_\pm)|(j_1m_1)(j_2m_2)\rangle &= \sqrt{(j_1 \mp m_1)(j_1 \pm m_1 + 1)}|(j_1m_1 \pm 1)(j_2m_2)\rangle \\ &\quad + \sqrt{(j_2 \mp m_2)(j_2 \pm m_2 + 1)}|(j_1m_1)(j_2m_2 \pm 1)\rangle.\end{aligned} \tag{5.5}$$

Using (5.3), we obtain

$$\begin{aligned}Z_\pm|\varphi; m_1m_2\rangle &= \sqrt{(j_1 \mp m_1)(j_1 \pm m_1 + 1)}|\varphi; m_1 \pm 1m_2\rangle \\ &\quad + \sqrt{(j_2 \mp m_2)(j_2 \pm m_2 + 1)}|\varphi; m_1m_2 \pm 1\rangle\end{aligned} \tag{5.6}$$

where $|\varphi; m_1m_2\rangle$ is the vector obtained from (5.4) by a contraction:

$$|\varphi; m_1m_2\rangle = \sum_{k_i=-j_i}^{j_i} \alpha^{m_1,m_2}_{k_1,k_2} t_{j_1k_1}|j_2k_2\rangle.$$

The inner product of $|j\ m \pm 1\rangle$ and (5.6) gives the recurrence relations for $\langle jm|\varphi; m_1m_2\rangle$

$$\begin{aligned}\sqrt{(j \mp m)(j \pm m + 1)}\langle jm|\varphi; m_1m_2\rangle &= \sqrt{(j_1 \mp m_1)(j_1 \pm m_1 + 1)}\langle j\ m \pm 1|\varphi; m_1 \pm 1m_2\rangle \\ &\quad + \sqrt{(j_2 \mp m_2)(j_2 \pm m_2 + 1)}\langle j\ m \pm 1|\varphi; m_1m_2 \pm 1\rangle.\end{aligned} \tag{5.7}$$

The recurrence relations (5.7) are the same as those for the $sl(2)$ CGCs; therefore, the quantity $\langle jm|\varphi; m_1m_2\rangle$ must be proportional to the $sl(2)$ CGCs. Denoting the proportional coefficient by $I(j_1j_2j)$,

$$\langle jm|\varphi; m_1m_2\rangle = \sum_{k_i=-j_i}^{j_i} \alpha^{m_1,m_2}_{k_1,k_2}\langle jm|t_{j_1k_1}|j_2k_2\rangle = C^{j_1,j_2,j}_{m_1,m_2,m} I(j_1j_2j). \tag{5.8}$$

This relation can be solved with respect to $\langle jm|t_{j_1k_1}|j_2k_2\rangle$, and the Wigner–Eckart theorem (5.1) has been proved. □

Remark. From (3.7) and the fact that the $sl(2)$ CGCs for bra and ket vectors are equal, we see that the quantity appearing on the right-hand side of (5.1) is the $\mathcal{U}_h(sl(2))$ CGC for bra vectors. This is a general property of the Wigner–Eckart theorem [18].

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